



Super edge-magic labeling of graphs: deficiency and maximality

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Abstract

A graph G of order p and size q is called *super edge-magic* if there exists a bijective function f from $V(G) \cup E(G)$ to $\{1, 2, 3, \dots, p + q\}$ such that $f(x) + f(xy) + f(y)$ is a constant for every edge $xy \in E(G)$ and $f(V(G)) = \{1, 2, 3, \dots, p\}$. The *super edge-magic deficiency* of a graph G is either the smallest nonnegative integer n such that $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n . In this paper, we study the super edge-magic deficiency of join product graphs. We found a lower bound of the super edge-magic deficiency of join product of any connected graph with isolated vertices and a better upper bound of the super edge-magic deficiency of join product of super edge-magic graphs with isolated vertices. Also, we provide constructions of some maximal graphs, ie. super edge-magic graphs with maximal number of edges.

Keywords: super edge-magic graph, super edge magic deficiency, join product graph, maximal graph

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1. Introduction

Let G be a finite and simple graph of order $|V(G)|$ and size $|E(G)|$. A graph G is *edge-magic* if there exists a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, |V(G)| + |E(G)|\}$ such that $f(x) + f(xy) + f(y) = k$ is a constant, called the *magic constant* of f , for every

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edge xy of G . In such a case f is called an *edge-magic labeling* of G . The concepts of edge-magic graphs and edge-magic labelings introduced by Kotzig and Rosa in 1970 [11]. Motivated by the concept of edge-magic labelings, Enomoto, Lladó, Nakamigawa, and Ringel [5] introduced the concepts of super edge-magic labelings and super edge-magic graphs as follows: A *super edge-magic labeling* of a graph G is an edge-magic labeling f of G with the extra condition that $f(V(G)) = \{1, 2, 3, \dots, |V(G)|\}$. A graph having a super edge-magic labeling is a *super edge-magic graph*. In [6], Figueroa-Centeno *et al.* provided a useful characterization of a super edge-magic graph that we state in the following lemma.

Lemma 1.1. [6] *A graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, \dots, |V(G)|\}$ such that the set $S = \{f(x) + f(y) : xy \in E(G)\}$ is a set of $|E(G)|$ consecutive integers.*

In light of above result, it suffices to exhibit the vertex labeling of a super edge-magic graph. The next lemma proved by Enomoto *et al.* [5] gives sufficient condition for un-existence of super edge-magic labeling of a graph.

Lemma 1.2. [5] *If G is a super edge-magic graph, then $|E(G)| \leq 2|V(G)| - 3$.*

In [11], Kotzig and Rosa proved that for every graph G there exists a nonnegative integer n such that $G \cup nK_1$ is edge-magic. This fact motivated them to introduced the concept of edge-magic deficiency of a graph. The *edge-magic deficiency* of a graph G , $\mu(G)$, is the smallest nonnegative integer n such that $G \cup nK_1$ is an edge-magic graph. Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa-Centeno *et al.* [7] introduced the concept of super edge-magic deficiency of a graph. The *super edge-magic deficiency* of a graph G , $\mu_s(G)$, is either the smallest nonnegative integer n such that $G \cup nK_1$ is a super edge-magic graph or $+\infty$ if there exists no such n .

Some papers concerning on the super edge-magic deficiency of graphs have been published. The super edge-magic deficiency of cycles, complete graphs, complete bipartite graphs $K_{2,m}$, disjoint union of cycles, and some forest with two components can be found in [7, 8]. The super edge-magic deficiency of some classes of chain graphs, wheels, fans, double fans, and disjoint union of complete bipartite graphs can be found in [13, 14]. The super edge-magic deficiency of complete bipartite graphs $K_{m,n}$ and disjoint union of stars are investigated in [10]. The super edge-magic deficiency of some classes of unicyclic graphs are studied in [1, 2]. The latest developments in these and other types of graph labelings can be found in [9].

2. Super edge-magic deficiency of join product graphs

Join product of two graphs is their graph union with additional edges that connect all vertices of the first graph to each vertex of the second graph. The join product of graphs G and H is denoted by $G + H$. Hence,

$$V(G + H) = V(G) \cup V(H)$$

and

$$E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}.$$

Ngurah and Simanjuntak [12] studied super edge-magic deficiency of the join product of a path, a star, and a cycle, respectively, with isolated vertices. We gave the lower and upper bounds of super edge-magic deficiency of $P_n + mK_1$, $K_{1,n} + mK_1$, and $C_n + mK_1$, respectively. We also gave the following result.

Lemma 2.1. [12] *For any integer $m \geq 1$, $\mu_s(P_4 + mK_1) = m - 1$ and $\mu_s(P_6 + mK_1) = 2(m - 1)$.*

In 2008, Ngurah *et al.* [14] studied super edge-magic deficiency of $P_n + 2K_1$. They have the following result and conjectured that for any odd integer n , $\mu_s(P_n + 2K_1) = \frac{n-1}{2}$.

Theorem 2.1. [14] *For any even integer $n \geq 2$, $\mu_s(P_n + 2K_1) = \frac{n-2}{2}$.*

In this section, we study super edge-magic deficiency of join product of any connected graphs with isolated vertices. Notice that the only connected graphs of order 1 and 2 are K_1 and K_2 , respectively. It is well known that $K_1 + mK_1$ and $K_2 + mK_1$ are super edge-magic for any integers $m \geq 1$. In other words, for any integers $m \geq 1$, $\mu_s(K_1 + mK_1) = \mu_s(K_2 + mK_1) = 0$. Here, we study super edge-magic deficiency of $G + mK_1$ where G is any connected graph with $|V(G)| \geq 3$. Our first result provides a lower bound of its super edge-magic deficiency.

Lemma 2.2. *If G is a connected graphs with $|V(G)| \geq 3$ and $m \geq 2$, then $\mu_s(G + mK_1) \geq \lfloor \frac{1}{2}(|E(G)| + (m - 2)|V(G)|) \rfloor - (m - 2)$ and this bound is tight.*

Proof. It is clear that $G + mK_1$ is a graph of order $|V(G)| + m$ and size $|E(G)| + m|V(G)| \geq (m + 1)|V(G)| - 1$. By Lemma 1.2, $G + mK_1$ is not super edge-magic. Again, by Lemma 1.2, the graph $(G + mK_1) \cup tK_1$, where $t = \lfloor \frac{1}{2}(|E(G)| + (m - 2)|V(G)|) \rfloor - (m - 1)$, is not a super edge-magic graph. Hence, we have the desire result.

The bound provides $\mu_s(P_4 + mK_1) \geq m - 1$, $\mu_s(P_6 + mK_1) \geq 2(m - 1)$, and $\mu_s(P_n + 2K_1) \geq \frac{n-2}{2}$. Base on Lemma 2.1 and Theorem 2.1, the lower bound is tight. \square

In [12], Ngurah and Simanjuntak also showed the finiteness of the super edge-magic deficiency of the join product of any super edge-magic graph with isolated vertices. We proved the following result.

Theorem 2.2. [12] *Let G be a super edge-magic graph of order p with a super edge-magic labeling f . For any integer $m \geq 1$, $\mu_s(G + mK_1) \leq \max(S) + (m - 2)p - m$, where $S = \{f(u) + f(v) : uv \in E(G)\}$.*

In the next theorem, we provide a better upper bound than the one in Theorem 2.2.

Theorem 2.3. *Let G be a super edge-magic graph of order p and size $q \geq 1$ with a super edge-magic labeling f . For any integer $m \geq 1$,*

$$\mu_s(G + mK_1) \leq \begin{cases} p + 1 - \min(S), & \text{if } m = 1, \\ (m - 2)(p - 1) + (q - 1), & \text{if } m \geq 2, \end{cases}$$

where $S = \{f(x) + f(y) : xy \in E(G)\}$.

Proof. By Lemma 1.1, S is a set of q consecutive integers. Define $H = G + mK_1$, $m \geq 1$, as a graph with

$$V(H) = V(G) \cup \{v_1, v_2, \dots, v_m\}$$

and

$$E(H) = E(G) \cup \{xv_i : x \in V(G), 1 \leq i \leq m\}.$$

Next, define an injective function g of H as follows.

$$g(u) = \begin{cases} f(u) + \alpha, & \text{if } u = x, x \in V(G), \\ 1, & \text{if } u = v_1, \\ \alpha + \max(S) + (i - 2)p, & \text{if } u = v_i, 2 \leq i \leq m, \end{cases}$$

where $\alpha = p + 2 - \min(S)$.

Next, we consider two cases depending on the value of m .

Case 1: $m = 1$.

It is easy to verify that $S_0 = \{g(x) + g(y) : xy \in E(G)\} = 2\alpha + S = \{2\alpha + \min(S), 2\alpha + \min(S) + 1, \dots, 2\alpha + \max(S)\}$ and $S_1 = \{g(v_1) + g(x) : x \in V(G)\} = \{\alpha + 2, \alpha + 3, \dots, \alpha + p + 1\}$. Also, it can be checked that $\max(S_1) + 1 = \min(S_0)$. Thus, $\{g(u) + g(v) : uv \in E(H)\} = S_1 \cup S_0$ is a set of $p + q$ consecutive integers. We can also check that the largest vertex label used is $p + \alpha$. So, there exist $p + \alpha - (p + 1) = p + 1 - \min(S)$ labels are not utilized. For each number between 1 and $p + \alpha$ that has not been used as a label, we introduce a new isolated vertex labeled with that number. Hence, by Lemma 1.1, g can be extended to a super edge magic labeling of $H \cup (p + 1 - \min(S))K_1$ with magic constant $3\alpha + p + 1 + \max(S)$. Thus, $\mu_s(G + K_1) \leq p + 1 - \min(S)$.

Case 2: $m \geq 2$.

It can be checked that under the labeling g , $S_i = \{g(v_i) + g(x) : x \in V(G)\} = \{2\alpha + \max(S) + (i - 2)p + 1, 2\alpha + \max(S) + (i - 2)p + 2, \dots, 2\alpha + \max(S) + (i - 1)p\}$, $2 \leq i \leq m$. Also, it can be checked that $\max(S_0) + 1 = \min(S_2)$, and for $2 \leq i \leq m - 1$, $\max(S_i) + 1 = \min(S_{i+1})$. Hence, $\{g(u) + g(v) : uv \in E(H)\} = S_1 \cup S_0 \cup \bigcup_{i=2}^m S_i$ is a set of $mp + q$ consecutive integers. The largest vertex label used is $\alpha + \max(S) + (m - 2)p$. By a similar argument as in Case 1, g can be extended to a super edge-magic labeling of $H \cup ((m - 2)(p - 1) + (q - 1))K_1$ with magic constant $3\alpha + 2\max(S) + (m - 1)p + (m - 2)p + 1$. Thus, $\mu_s(G + mK_1) \leq (m - 2)(p - 1) + (q - 1)$. \square

By combining Lemma 2.2 and Theorem 2.3, the super edge-magic deficiency of join product of a path, a star, or a cycle with isolated vertices can be stated as follows (also see [12]).

Corollary 2.1. *Let $m \geq 2$ be an integer.*

- For any integers $n \geq 3$, the super edge-magic deficiency of $P_n + mK_1$ satisfies $\lfloor \frac{1}{2}(n(m - 1) - 1) \rfloor - (m - 2) \leq \mu_s(P_n + mK_1) \leq (m - 1)(n - 1) - 1$.
- For any integers $n \geq 2$, the super edge-magic deficiency of $K_{1,n} + mK_1$ satisfies $\lfloor \frac{1}{2}((n + 1)(m - 1) - 1) \rfloor - (m - 2) \leq \mu_s(K_{1,n} + mK_1) \leq n(m - 1) - 1$.
- For any odd integers $n \geq 3$, the super edge-magic deficiency of $C_n + mK_1$ satisfies $\lfloor \frac{1}{2}n(m - 1) \rfloor - (m - 2) \leq \mu_s(C_n + mK_1) \leq (n - 1)(m - 1)$.

3. Constructions of maximal graphs

As a consequence of Lemma 1.2, maximal number of edges of a super edge-magic graph G is $2|V(G)| - 3$. A super edge-magic graph G which satisfy $|E(G)| = 2|V(G)| - 3$ is called *maximal*. In this section, we seek for some maximal graphs. First, we study the maximality of graphs obtained from chain graphs by adding or deleting an edge of the chain graphs. A *chain graph* with blocks $B_1, B_2, B_3, \dots, B_k$, denoted by $C[B_1, B_2, \dots, B_k]$, is defined as a graph such that for every i , B_i and B_{i+1} have a common vertex in such a way that the block-cut-vertex graph is a path. The concept of a chain graph was firstly introduced by Barrientos in 2002 [3]. If $B_i = H$, for every i , then $C[B_1, B_2, \dots, B_k]$ denoted by kH -path. Hence, kK_4 -path is a chain graph with k blocks where each block is identical and isomorphic to the complete graph K_4 . If c_1, c_2, \dots, c_{k-1} are the consecutive cut vertices of $C[B_1, B_2, \dots, B_k]$, then $(k-2)$ -tuple $(d_1, d_2, \dots, d_{k-2})$ where d_i is the distance between c_i and c_{i+1} , $1 \leq i \leq k-2$ is called the *string* of $C[B_1, B_2, \dots, B_k]$.

For every integer $k \geq 3$, let $H = kK_4$ -path be a graph with vertex set and edge set as follows.

$$V(H) = \{x_i, y_i : 1 \leq i \leq k\} \cup \{c_i : 1 \leq i \leq k+1\}$$

and

$$E(H) = \{c_i c_{i+1}, c_i x_i, c_i y_i, x_i y_i, x_i c_{i+1}, y_i c_{i+1} : 1 \leq i \leq k\}.$$

As we can see, H is a graph of order $3k+1$ and size $6k$. Ngurah et al. [13] proved that $\mu_s(H) = 1$. The following vertex labeling f is an alternative vertex labeling that extends to a super edge-magic labeling of $H \cup K_1$.

$$f(u) = \begin{cases} 3i-2, & \text{if } u = x_i, 1 \leq i \leq k, \\ 3i-1, & \text{if } u = c_i, 1 \leq i \leq k+1, \\ 3i, & \text{if } u = y_i, 1 \leq i \leq k, \\ 3k+1, & \text{if } u = K_1. \end{cases}$$

From now on f refers to this vertex labeling.

It is clear that by removing one edge from H , the resulting graph has number of edges satisfying the maximal condition. We shall study two of such graphs. Let j be an integer where $1 \leq j \leq k$ and let $H_{[j]} = H - \{x_j y_j\}$ and $H^{[j]} = H - \{c_j c_{j+1}\}$. It is clear that for every $1 \leq j \leq k$, $H_{[j]}$ and $H_{[k+1-j]}$ (resp. $H^{[j]}$ and $H^{[k+1-j]}$) are isomorphic,. So, it is suffices to study maximality of $H_{[j]}$ (resp. $H^{[j]}$) for $1 \leq j \leq \lceil \frac{k}{2} \rceil$.

Theorem 3.1. *For every integers $k \geq 3$ and $1 \leq j \leq \lceil \frac{k}{2} \rceil$, $H_{[j]}$ and $H^{[j]}$ are maximal.*

Proof. Define a labeling g as follows.

For case $j = 1$,

$$g(u) = \begin{cases} f(u), & \text{if } u = c_1, x_1, \\ f(x_{i+1}), & \text{if } u = x_i, 2 \leq i \leq k-1, \\ f(K_1), & \text{if } u = x_k, \\ f(x_2), & \text{if } u = y_1, \\ f(y_{i-1}), & \text{if } u = c_i, 2 \leq i \leq k, \\ f(c_i), & \text{if } u = y_i, 2 \leq i \leq k. \end{cases}$$

For case $j = 2, 3, \dots, \lceil \frac{k}{2} \rceil$,

$$g(u) = \begin{cases} f(u), & \text{if } u = c_i, x_i, 1 \leq i \leq j, \\ f(u), & \text{if } u = y_i, 1 \leq i \leq j-1, \\ f(x_{i+1}), & \text{if } u = x_i, j \leq i \leq k-1, \\ f(K_1), & \text{if } u = x_k, \\ f(x_{j+1}), & \text{if } u = y_j, \\ f(y_{i-1}), & \text{if } u = c_i, j+1 \leq i \leq k+1, \\ f(c_i), & \text{if } u = y_i, j+1 \leq i \leq k. \end{cases}$$

It can be checked that g is a bijection from $V(H_{[j]})$ to $\{1, 2, 3, \dots, 3k+1\}$ and for $1 \leq j \leq \lceil \frac{k}{2} \rceil$, $\{g(u) + g(v) : uv \in E(H_{[j]})\} = \{3, 4, 5, \dots, 6k+1\}$. By Lemma 1.1, g can be extended to a super edge-magic labeling of $H_{[j]}$. Also, it is easy to verify that for $1 \leq j \leq \lceil \frac{k}{2} \rceil$, $g(x_j) + g(y_j) = g(c_j) + g(c_{j+1})$. Hence, g can be extended to a super edge-magic labeling of $H^{[j]}$. \square

Now, we consider $H - \{c_1x_1\}$ and $H - \{x_1c_2\}$.

Theorem 3.2. For every integer $k \geq 3$, $H - \{c_1x_1\}$ and $H - \{x_1c_2\}$ are maximal.

Proof. Define a labeling of $H - \{x_1c_2\}$ and $H - \{c_1x_1\}$ as follows.

$$h(u) = \begin{cases} f(x_1), & \text{if } u = c_1, \\ f(c_1), & \text{if } u = x_1, x_1 \in V(H - \{x_1c_2\}), \\ f(x_2), & \text{if } u = y_1, y_1 \in V(H - \{x_1c_2\}), \\ f(c_1), & \text{if } u = y_1, y_1 \in V(H - \{c_1x_1\}), \\ f(x_2), & \text{if } u = x_1, x_1 \in V(H - \{c_1x_1\}), \\ f(K_1), & \text{if } u = x_k, \\ f(x_{i+1}), & \text{if } u = x_i, 2 \leq i \leq k-1, \\ f(K_1), & \text{if } u = x_k, \\ f(y_i), & \text{if } u = c_{i+1}, 1 \leq i \leq k, \\ f(c_i), & \text{if } u = y_i, 2 \leq i \leq k. \end{cases}$$

It is a routine procedure to verify that h can be extended to a super edge-magic labeling of $H - \{x_1c_1\}$ and $H - \{x_1c_2\}$. \square

Notice that for every $1 \leq i \leq k$, $H - \{c_ix_i\}$ and $H - \{x_{k+1-i}c_{k+2-i}\}$ (resp. $H - \{x_ic_{i+1}\}$ and $H - \{x_{k+1-i}c_{k+1-i}\}$) are isomorphic. Also, it can be checked that $H - \{c_ix_i\}$ and $H - \{c_iy_i\}$ (resp. $H - \{x_ic_{i+1}\}$ and $H - \{y_ic_{i+1}\}$) are isomorphic. Referring to these facts and the afore-mentioned results, we propose the following problems.

Problem 1. For every integers $k \geq 3$ and $2 \leq j \leq \lceil \frac{k}{2} \rceil$, study the super edge-magicness of $H - \{c_jx_j\}$ and $H - \{x_jc_{j+1}\}$.

Next, consider the graph $F[k]$, where $F[k]$ is a graph obtained from the graph $H = kK_4$ -path by attaching a pendant vertex z to the vertex c_{k+1} of H . Hence, $F[k]$ is chain graph with $k+1$ blocks where the first k blocks are K_4 and the last block is a K_2 . It can be checked that the labeling $h(x) = f(x)$ for every vertex $x \in V(H)$ and $h(z) = 3k+1$ can be extended to a super edge-magic labeling of $F[k]$. Thus $F[k]$ is a maximal graph. Hence, we have the following result.

Theorem 3.3. For every integer $k \geq 3$, $F[k]$ is maximal.

Next, we investigate the maximality of chain graphs $G(k) = C[B_1, B_2, \dots, B_k]$ which satisfy the following conditions. i). If $k \geq 3$ is an odd integer, then B_1, B_3, \dots, B_k are one of the graphs in Figure 1 (a) and B_2, B_4, \dots, B_{k-1} are the graph in Figure 1 (b). ii). If $k \geq 4$ is an even integer, then B_1, B_3, \dots, B_{k-1} are one of the graphs in Figure 1 (a), B_2, B_4, \dots, B_{k-2} are the graph in Figure 1 (b), and B_k is one of the graphs in Figure 1 (c). The chain graphs $G(k) = C[B_1, B_2, \dots, B_k]$ are constructed from blocks B_1, B_2, \dots, B_k by identifying vertex $x_{3,j}$ and vertex $y_{1,j+1}$, $1 \leq j \leq k-1$, and we denote these new vertices by c_1, c_2, \dots, c_{k-1} . Thus, $G(k)$ is a graph of order $5k + 1$ and size $10k - 1$. The string of $G(k)$ is $(d_1, d_2, \dots, d_{k-2})$ where $d_i = 2$ when i is odd and $d_i = 2$ or 3 when i is even. We shall prove that $G(k)$ is maximal.

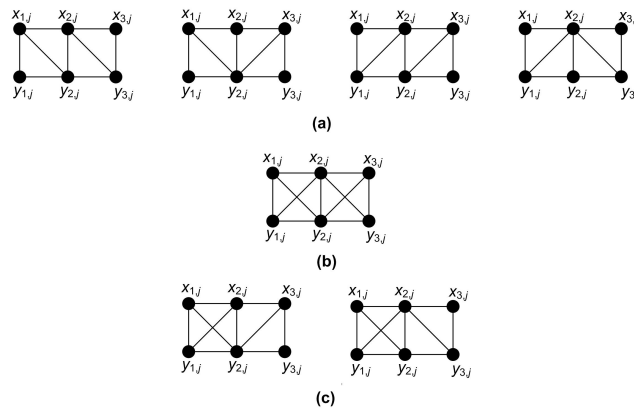


Figure 1. The blocks of the graphs $G(k) = C[B_1, B_2, B_3, \dots, B_k]$.

Theorem 3.4. For every integer $k \geq 3$, $G(k)$ is maximal.

Proof. It is not hard to verify that the following labeling is a super edge-magic labeling of $G(k)$.

$$h(u) = \begin{cases} (1, 3, 2, 4, 6), & \text{if } u = (x_{1,1}, x_{2,1}, y_{1,1}, y_{2,1}, y_{3,1}), \\ (7, 9, 8, 10), & \text{if } u = (x_{1,2}, x_{2,2}, y_{2,2}, y_{3,2}), \\ 5, & \text{if } u = c_1, \\ 12, & \text{if } u = c_2, \\ 5 + 10j, & \text{if } u = c_{2j+1}, 1 \leq j \leq \lfloor \frac{k-2}{2} \rfloor, \\ 12 + 10j, & \text{if } u = c_{2j+2}, 1 \leq j \leq \lfloor \frac{k-3}{2} \rfloor, \\ h(x_{i,1}) + 10j, & \text{if } u = x_{i,2j+1}, 1 \leq i \leq 2, 1 \leq j \leq \lfloor \frac{k-1}{2} \rfloor, \\ h(y_{i,1}) + 10j, & \text{if } u = y_{i,2j+1}, 2 \leq i \leq 3, 1 \leq j \leq \lfloor \frac{k-1}{2} \rfloor, \\ h(x_{i,2}) + 10j, & \text{if } u = x_{i,2j+1}, 1 \leq i \leq 2, 1 \leq j \leq \lfloor \frac{k-2}{2} \rfloor, \\ h(y_{i,2}) + 10j, & \text{if } u = y_{i,2j+1}, 2 \leq i \leq 3, 1 \leq j \leq \lfloor \frac{k-2}{2} \rfloor, \\ 5k, & \text{if } u = y_{3,k}, k \text{ is odd}, \\ 5k + 1, & \text{if } u = y_{3,k}, k \text{ is even}. \end{cases}$$

□

Next, we consider graphs obtaining from ladders $L_m = P_m \times P_2$, $m \geq 2$, by adding a diagonal in each rectangle of L_m . We denote such a graph by $G[m]$. Hence, $G[m]$ can be defined as a graph with the vertex and edge sets as follows.

$$V(G[m]) = \{u_j, v_j | 1 \leq j \leq m\}$$

$$E(G[m]) = \{u_j v_j | 1 \leq j \leq m\} \cup \{u_j u_{j+1}, v_j v_{j+1} | 1 \leq j \leq m-1\} \cup \mathbb{S},$$

where $\mathbb{S} = \{e_i : e_i \text{ is either } u_j v_{j+1} \text{ or } v_j u_{j+1}, 1 \leq i \leq m-1\}$. It can be checked that $|E(G[m])| = 2|V(G[m])| - 3$.

Theorem 3.5. *For any integer $m \geq 2$, $G[m]$ is maximal.*

Proof. It is easy to verify that the bijection $h : V(G[m]) \rightarrow \{1, 2, \dots, 2m\}$ defined by

$$h(x) = \begin{cases} 2i, & \text{if } x = u_j, 1 \leq j \leq m, \\ 2i - 1, & \text{if } x = v_j, 1 \leq j \leq m, \end{cases}$$

can be extended to a super edge-magic labeling of $G[m]$ with magic constant $6m$. □

Our results presented in Theorems 3.1, 3.2, 3.3, 3.4, and 3.5, give impression that graphs satisfy the maximal condition are super edge-magic. However, it is not the case. As an example $P_n + K_1$, which satisfies the maximal condition, is super edge-magic if and only if $1 \leq n \leq 6$ (see [7]).

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